

PROPERTIES OF THE DOT PRODUCT GRAPH OF A COMMUTATIVE RING

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ABSTRACT. Let R be a commutative ring with identity and $n \geq 1$ be an integer. Let $R^n = R \times \cdots \times R$ (n times). The *total dot product* graph, denoted by $TD(R, n)$ is a simple graph with elements of $R^n - \{(0, 0, \dots, 0)\}$ as vertices, and two distinct vertices \mathbf{x} and \mathbf{y} are adjacent if and only if $\mathbf{x} \cdot \mathbf{y} = 0 \in R$, where $\mathbf{x} \cdot \mathbf{y}$ denotes the dot product of \mathbf{x} and \mathbf{y} . In this paper, we find the structure of $TD(R \times S, n)$ with respect to the structure of $TD(R, n)$ and $TD(S, n)$. In addition, we find the degree of vertices of this graph. We determine when it is regular. Let \mathbb{F} be a finite field. It is shown that if $TD(\mathbb{F}, n) \simeq TD(R, m)$, then $n = m$ and $R \simeq \mathbb{F}$. A number of results concerning the domination number are also presented. Furthermore, we give some results on the clique and the independence number of $TD(R, n)$. It is shown that the ring R is finite if and only if its independence number is finite. Finally, we classify all planar graphs within this class.

1. INTRODUCTION

There are many papers purporting to study the interplay between commutative rings and combinatorics typically, these involve starting with a ring and studying some graph associated to it (e.g. zero-divisor graph, unitary Cayley graph). By virtue of their definition, most of these graphs have a lot of symmetry, and hence lend themselves well to the computation of various combinatorial invariants; this pursuit has attracted the attention of many people in the last three decades, see [2, 3, 5, 6, 7, 12, 13, 14].

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Let R be a commutative ring with nonzero identity and $n \geq 1$ be an integer. Let $R^n = R \times \cdots \times R$ (n times). Badawi [5] introduced the *total dot product* graph, denoted by $TD(R, n)$, as a simple graph with elements of $R^n - \{(0, 0, \dots, 0)\}$ as vertices, and two distinct vertices \mathbf{x} and \mathbf{y} are adjacent if and only if $\mathbf{x} \cdot \mathbf{y} = 0 \in R$, where $\mathbf{x} \cdot \mathbf{y}$ denotes the dot product of \mathbf{x} and \mathbf{y} . For example, figure (1) depicts $TD(\mathbb{Z}_2, 3)$. In [5], it was shown that the diameter of this graph for $n \geq 3$ is 3. Also, for $n = 2$, the diameter was determined. In addition, the girth of this graph was studied.

By the *zero-divisor graph* $\Gamma(R)$ of R , we mean the graph with vertices $Z(R) - \{0\}$ such that there is an (undirected) edge between vertices a and b if and only if $a \neq b$ and $ab = 0$. For an arbitrary natural number n and ring R , it can be easily seen that there exist n mutually distinct copies of $\Gamma(R)$ in $TD(R, n)$.

Throughout this paper, we use $N(v)$ for the neighborhood of a vertex (i.e. the set of vertices adjacent to v). For a graph G , let $V(G)$ denote the set of vertices. The *tensor product* of G_1 and G_2 , $G_1 \otimes G_2$, is the graph with vertex set $V(G_1 \otimes G_2) := V(G_1) \times V(G_2)$, specified by putting (u, v) adjacent to (u', v') if and only if u is adjacent to u' in G_1 and v is adjacent to v' in G_2 .

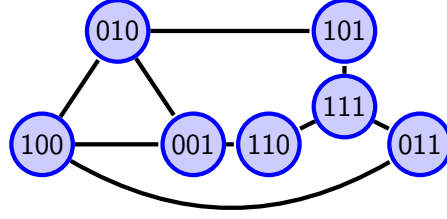
A set D of vertices of a graph G is said to be *dominating* if every vertex of $V(G) - D$ is adjacent to a vertex of D , and the *domination number* $\gamma(G)$ is the minimum number of vertices of a dominating set in G . For a given graph G and a natural number k , the decision problem testing whether $\gamma(G) \leq k$ was shown to be NP-complete [8].

A subset I of $V(G)$ is said to be *independent* if any two vertices in that subset are pairwise non-adjacent. The *independence number* of a graph G , denoted by $\alpha(G)$, is the maximum size of an independent set of vertices in G .

A *clique* is a set of pairwise adjacent vertices in a graph. The *clique number* of a graph G , denoted by $\omega(G)$, is the size of the largest clique of G .

A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. For basic terminology regarding graphs, we refer the reader to [17].

Throughout this paper, R is a finite commutative ring with identity. Here R^* and $U(R)$ stand for $R - \{0\}$ and invertible elements of R , respectively. A ring R is said to be *reduced* if R has no nonzero nilpotent element. So, a finite commutative reduced ring R is a finite product of finite

FIGURE 1. $TD(\mathbb{Z}_2, 3)$

fields. Let $\mathbf{a} = (a_1, \dots, a_n) \in R^n$, by $\|\mathbf{a}\|$ we denote $a_1^2 + a_2^2 + \dots + a_n^2$. We shall denote by \mathbb{Z}_n the ring of integers modulo n . Let e_i ($i = 1, \dots, n$) be the element in R^n such that j -coordinate is 0 for $j \neq i$, and i -coordinate is 1.

Let $\overline{TD}(R, n)$ be the graph whose vertex set is R^n , and in which \mathbf{x} is adjacent to \mathbf{y} if and only if $\mathbf{x} \cdot \mathbf{y} = 0$. Therefore we have loops. Let $G = \overline{TD}(R, n)$. Remove the vertex with maximum degree and vertices with loops, so the new graph is $\overline{TD}(R, n)$. Thus, $\overline{TD}(R, n)$ and $TD(R, n)$ have a lot of similarities. Then, it is worthwhile to study $\overline{TD}(R, n)$. Figure (2) shows $\overline{TD}(\mathbb{Z}_2, 3)$.

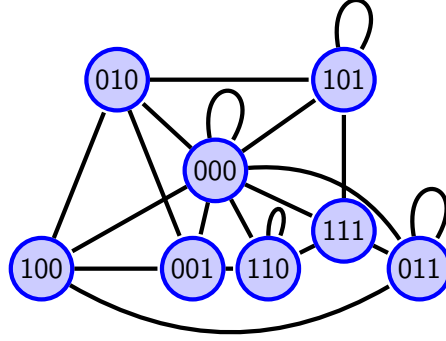
In section 2, we proceed with the study of the graph $\overline{TD}(R, n)$. In addition, we study the degree of vertices in $TD(R, n)$. Finally, we prove that if $TD(\mathbb{F}, n) \simeq TD(R, m)$, where \mathbb{F} is a finite field, then $n = m$ and $R \simeq \mathbb{F}$. Section 3 is devoted to the study of the domination number of $TD(R, n)$. We find the domination number of $TD(\mathbb{F}, n)$. We give some upper bounds for an arbitrary ring. Moreover, we will discuss the domination number of $TD(R, n)$ for infinite rings. In the fourth section, we will look at the clique and independence number. The last section in this paper lists all planar graphs within this class.

2. DEGREE SEQUENCE AND $\overline{TD}(R, n)$

It is natural to relate $\overline{TD}(R \times S, n)$ to $\overline{TD}(R, n)$ and $\overline{TD}(S, n)$. The first theorem provides the relation between these graphs.

Theorem 2.1. *Let R and S be arbitrary rings. Then $\overline{TD}(R \times S, n) \simeq \overline{TD}(R, n) \otimes \overline{TD}(S, n)$.*

Proof. Let $G = \overline{TD}(R \times S, n)$. The vertex $\mathbf{a} = ((r_1, s_1), (r_2, s_2), \dots, (r_n, s_n))$ in G is adjacent to $\mathbf{b} = ((r'_1, s'_1), (r'_2, s'_2), \dots, (r'_n, s'_n))$ if and only if $\mathbf{a} \cdot \mathbf{b} = (\sum_{i=1}^n r_i r'_i, \sum_{i=1}^n s_i s'_i) = (0, 0)$. Equivalently, $\mathbf{r} = (r_1, r_2, \dots, r_n)$ is adjacent to $\mathbf{r}' = (r'_1, r'_2, \dots, r'_n)$ in $\overline{TD}(R, n)$ and $\mathbf{s} = (s_1, s_2, \dots, s_n)$ is adjacent to $\mathbf{s}' = (s'_1, s'_2, \dots, s'_n)$ in $\overline{TD}(S, n)$, which proves the theorem. \square

FIGURE 2. $\overline{TD}(\mathbb{Z}_2, 3)$.

If R is a finite commutative ring, then $R \simeq R_1 \times \cdots \times R_t$ where each R_i is a finite commutative local ring with maximal ideal M_i , by Theorem 8.7 of [4]. Hence, by the aforementioned theorem, $\overline{TD}(R, n) \simeq \bigotimes_{i=1}^t \overline{TD}(R_i, n)$.

Remark 1. Let R and S be arbitrary rings. Theorem 2.1 immediately tells us that the number of loops in $\overline{TD}(R \times S, n)$ is product of the number loops of $\overline{TD}(R, n)$ and $\overline{TD}(S, n)$.

Remark 2. Let $O(R, n)$ be the number of non-trivial solutions of the equation $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$. Then the number of loops in $\overline{TD}(R, n)$ equals to $O(R, n) + 1$. By Exercise 19 in Chapter 8 of [10], we know that if n is an odd number and \mathbb{F} is the field of prime order p , then the number of loops in $\overline{TD}(\mathbb{F}, n)$ is p^{n-1} .

Let \mathbb{F} be a finite field of characteristic 2. Since $x_1^2 + x_2^2 + \cdots + x_n^2 = (x_1 + \cdots + x_n)^2$, it follows that the number of loops of $\overline{TD}(\mathbb{F}, n)$ is $|\mathbb{F}|^{n-1}$.

Theorem 2.2 (Chevalley–Warning). *Let \mathbb{F} be the field with $q = p^\alpha$ elements, where p is a prime number. If $f(x_1, \dots, x_n) \in \mathbb{F}[x_1, \dots, x_n]$ and $\deg(f) < n$, then $|\{(a_1, \dots, a_n) \in \mathbb{F}^n \mid f(a_1, \dots, a_n) = 0\}| \equiv 0 \pmod{p}$.*

Let $n > 2$. As a consequence of the Chevalley–Warning theorem, there exists a non-trivial loop in $\overline{TD}(\mathbb{F}, n)$. Moreover, the number of loops is divisible by the characteristic of \mathbb{F} .

It is readily known that the equation $x^2 + y^2 = 0$ has a non-trivial solution in \mathbb{F} if and only if $|\mathbb{F}| \not\equiv 3 \pmod{4}$. Therefore, we have:

$$O(\mathbb{F}, 2) = \begin{cases} 0 & |\mathbb{F}| \equiv 3 \pmod{4} \\ 2(|\mathbb{F}| - 1) & |\mathbb{F}| \equiv 1 \pmod{4} \\ |\mathbb{F}| - 1 & |\mathbb{F}| \text{ is even.} \end{cases}$$

In the remainder of this section, we will restrict our attention to the degree of vertices, and isomorphism problem for $TD(R, n)$. Let $\mathbf{a} \in R^n$ and $Z(\mathbf{a}) = \{\mathbf{b}; \mathbf{a} \cdot \mathbf{b} = 0\}$. Hence, $N(\mathbf{a}) = Z(\mathbf{a}) - \{0, \mathbf{a}\}$. Obviously, $Z(\mathbf{a})$ is a R -submodule of R^n , and $\deg \mathbf{a} = |N(\mathbf{a})| - 1$ if $\|\mathbf{a}\| \neq 0$ and $\deg \mathbf{a} = |N(\mathbf{a})| - 2$ otherwise.

In the following theorem, we will find the degree of the vertex $\mathbf{a} = (a_1, \dots, a_n) \in R^n$, if at least one coordinate is invertible.

Theorem 2.3. *Let R be a finite ring with nonzero identity. Let $\mathbf{a} = (a_1, \dots, a_n) \in R^n$ such that there exists $1 \leq i \leq n$ in such a way that a_i is invertible. Then for the degree of \mathbf{a} in the graph $TD(R, n)$, we have the following:*

- (a) *If $\|\mathbf{a}\| \neq 0$, then $\deg \mathbf{a} = |R|^{n-1} - 1$.*
- (b) *If $\|\mathbf{a}\| = 0$, then $\deg \mathbf{a} = |R|^{n-1} - 2$.*

Proof. There is no loss of generality in assuming that a_n is invertible. Thus, for any choice of x_1, x_2, \dots, x_{n-1} , there exists a unique $x_n \in R$ such that $\sum_{j=1}^{n-1} a_j x_j = -a_n x_n$. Then there exists exactly $|R|^{n-1}$ elements in R^n in such a way that $\mathbf{a} \cdot \mathbf{x} = 0$, which completes the proof. \square

Consequently, in the case of finite field, we conclude the following corollary.

Corollary 2.4. *Let \mathbb{F} be a finite field with q elements. Then the following hold:*

- (a) *If $n > 2$, then $TD(\mathbb{F}, n)$ is a semi-regular graph with degrees $q^{n-1} - 1$ and $q^{n-1} - 2$.*
- (b) *If $n = 2$ and $q \equiv 3 \pmod{4}$, then $TD(\mathbb{F}, n)$ is a regular graph of valency $q - 1$.*
- (c) *If $n = 2$ and $q \equiv 1 \pmod{4}$, then $TD(\mathbb{F}, n)$ is a semi-regular graph with degrees $q - 1$ and $q - 2$.*
- (d) *If $n = 2$ and $\text{char}(\mathbb{F}) = 2$, then $TD(\mathbb{F}, n)$ is a semi-regular graph with degrees $q - 1$ and $q - 2$.*

(e) If $n = 1$, then $TD(\mathbb{F}, n)$ is the empty graph with $q - 1$ vertices.

In [5] it was shown that if R is an integral domain, then $TD(R, 2)$ is disconnected. In the next theorem we find the structure of $TD(\mathbb{F}, 2)$.

Theorem 2.5. *Let \mathbb{F} be a finite field. Then $TD(\mathbb{F}, 2)$ is disconnected, and*

- (a) *If $O(\mathbb{F}, 2) = 0$, then the number of connected component is $\frac{|\mathbb{F}| + 1}{2}$. Moreover, $TD(\mathbb{F}, 2)$ is disjoint union of $\frac{|\mathbb{F}| + 1}{2}$ complete bipartite graphs $K_{|\mathbb{F}| - 1, |\mathbb{F}| - 1}$.*
- (b) *The graph $TD(\mathbb{F}, 2)$ is disjoint union of $\frac{O(2, \mathbb{F})}{|\mathbb{F}| - 1}$ complete graphs of size $|\mathbb{F}| - 1$ and $\frac{|\mathbb{F}|^2 - 1 - O(2, \mathbb{F})}{2(|\mathbb{F}| - 1)}$ complete bipartite graphs $K_{|\mathbb{F}| - 1, |\mathbb{F}| - 1}$.*

Proof. Let (a, b) be a vertex in $TD(\mathbb{F}, 2)$. We have two cases:

- (1) If $a^2 + b^2 \neq 0$. Let $A_1 = \{(ra, rb) \mid r \in \mathbb{F}^*\}$ and $A_2 = \{(-rb, ra) \mid r \in \mathbb{F}^*\}$. Obviously, the graph induced by $A_1 \cup A_2$ is isomorphic to the complete bipartite graph $K_{|\mathbb{F}| - 1, |\mathbb{F}| - 1}$. Then it is a connected component by Corollary 2.4.
- (2) If $a^2 + b^2 = 0$. Then the graph induced by $W = \{(ra, rb) \mid r \in \mathbb{F}^*\}$ is a clique of size $|\mathbb{F}| - 1$. Thus, it is a connected component by Corollary 2.4.

□

The next theorem shows that if \mathbb{F} and \mathbb{E} are different fields or $m \neq n$, then the graph $TD(\mathbb{F}, n)$ is not isomorphic to the graph $TD(\mathbb{E}, m)$.

Theorem 2.6. *Let \mathbb{F} and \mathbb{E} be finite fields, and let m, n be integers. If $TD(\mathbb{F}, n) \simeq TD(\mathbb{E}, m)$, then $m = n$ and $\mathbb{F} \simeq \mathbb{E}$.*

Proof. Let $|\mathbb{F}| = q$ and $|\mathbb{E}| = r$. The number of vertices of $TD(\mathbb{F}, n)$ and $TD(\mathbb{E}, m)$ are $q^n - 1$ and $r^m - 1$, respectively. Therefore,

$$(1) \quad q^n - 1 = r^m - 1.$$

The graphs $TD(\mathbb{F}, n)$ and $TD(\mathbb{E}, m)$ are regular or semi-regular graphs. The maximum degree of $TD(\mathbb{F}, n)$ and $TD(\mathbb{E}, m)$ are $q^{n-1} - 1$ and $r^{m-1} - 1$, respectively. Hence,

$$(2) \quad q^{n-1} - 1 = r^{m-1} - 1.$$

Combining equations (1) and (2), we can see that $n = m$ and $q = r$. \square

The next theorem deals with the degree of vertices for reduced rings.

Theorem 2.7. *Let $R = \mathbb{F}_1 \times \cdots \times \mathbb{F}_t$, where \mathbb{F}_i is a field for each $i = 1, \dots, t$. Then the degree of $\mathbf{a} = ((a_{11}, \dots, a_{1t}), \dots, (a_{n1}, \dots, a_{nt}))$ is*

$$\begin{cases} \frac{|R|^n}{\prod_{i=1}^t |\mathbb{F}_i|^{\tau_i}} - 1 & \text{if } \|\mathbf{a}\| \neq 0 \\ \frac{|R|^n}{\prod_{i=1}^t |\mathbb{F}_i|^{\tau_i}} - 2 & \text{if } \|\mathbf{a}\| = 0, \end{cases}$$

where,

$$\tau_i = \begin{cases} 0 & \text{if } (a_{1i}, a_{2i}, \dots, a_{ni}) = \mathbf{0} \\ 1 & \text{otherwise.} \end{cases}$$

In particular, the minimum degree of $TD(R, n)$ is either $|R|^{n-1} - 1$ or $|R|^{n-1} - 2$.

Proof. Let $\mathbf{a} = ((a_{11}, \dots, a_{1t}), \dots, (a_{n1}, \dots, a_{nt})) \in R^n$. Then $\mathbf{b} = ((b_{11}, \dots, b_{1t}), \dots, (b_{n1}, \dots, b_{nt}))$ is adjacent to \mathbf{a} if the following system of equations is satisfied:

$$\begin{cases} a_{11}b_{11} + a_{21}b_{21} + \cdots + a_{n1}b_{n1} = 0 \\ a_{12}b_{12} + a_{22}b_{22} + \cdots + a_{n2}b_{n2} = 0 \\ \vdots \\ a_{1t}b_{1t} + a_{2t}b_{2t} + \cdots + a_{nt}b_{nt} = 0. \end{cases}$$

Equations are independent, so the number of solutions is $\frac{\prod_{i=1}^t |\mathbb{F}_i|^n}{\prod_{i=1}^t |\mathbb{F}_i|^{\tau_i}}$, which completes the proof. \square

Remark 3. Let $\mathbf{g} = ((a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)) \in (R \times S)^n$. Let $\mathbf{a} = (a_1, \dots, a_n) \in R^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in S^n$. Let $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. Then the degree of \mathbf{g} in $TD(R \times S, n)$ is:

$$\begin{cases} (1 + \deg_R \mathbf{a})(1 + \deg_S \mathbf{b}) - 1 & \text{if } \|\mathbf{a}\| \neq 0 \text{ and } \|\mathbf{b}\| \neq 0 \\ (2 + \deg_R \mathbf{a})(1 + \deg_S \mathbf{b}) - 1 & \text{if } \|\mathbf{a}\| = 0 \text{ and } \|\mathbf{b}\| \neq 0 \\ (1 + \deg_R \mathbf{a})(2 + \deg_S \mathbf{b}) - 1 & \text{if } \|\mathbf{a}\| \neq 0 \text{ and } \|\mathbf{b}\| = 0 \\ (2 + \deg_R \mathbf{a})(2 + \deg_S \mathbf{b}) - 2 & \text{if } \|\mathbf{a}\| = 0 \text{ and } \|\mathbf{b}\| = 0, \end{cases}$$

where $\deg_R \mathbf{a}$ and $\deg_S \mathbf{b}$ denote the degree of \mathbf{a} and \mathbf{b} in $TD(R, n)$ and $TD(S, n)$, respectively.

If $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then the degree of \mathbf{g} in $TD(R \times S, n)$ is:

$$\begin{cases} |R|^n(1 + \deg_S \mathbf{b}) - 1 & \text{if } \|\mathbf{b}\| \neq 0 \\ |R|^n(2 + \deg_S \mathbf{b}) - 2 & \text{if } \|\mathbf{b}\| = 0. \end{cases}$$

If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, then the degree of \mathbf{g} in $TD(R \times S, n)$ is:

$$\begin{cases} |S|^n(1 + \deg_R \mathbf{a}) - 1 & \text{if } \|\mathbf{a}\| \neq 0 \\ |S|^n(2 + \deg_R \mathbf{a}) - 2 & \text{if } \|\mathbf{a}\| = 0. \end{cases}$$

Theorem 2.8. *Let $R = \mathbb{F}_1 \times \cdots \times \mathbb{F}_t$, $S = \mathbb{E}_1 \times \cdots \times \mathbb{E}_s$, where \mathbb{F}_i and \mathbb{E}_j are fields for each $i = 1, \dots, t$ and $j = 1, \dots, s$. Let m, n be integers. If $TD(R, n) \simeq TD(S, m)$. Then $m = n$ and $|R| = |S|$.*

Proof. Since $TD(R, n) \simeq TD(S, m)$, we have

$$|R|^n = |S|^m.$$

The minimum degree of $TD(R, n)$ is either $|R|^{n-1} - 1$ or $|R|^{n-1} - 2$. Also, the minimum degree of $TD(S, m)$ is either $|S|^{m-1} - 1$ or $|S|^{m-1} - 2$. Therefore, we can reduce to two cases:

(i) If

$$|R|^{n-1} - 1 = |S|^{m-1} - 2,$$

then we get

$$|S|^{m-1}(|R| - |S|) = |R|.$$

Thus,

$$|S|^{n(m-1)}(|R| - |S|)^n = |S|^m.$$

Hence,

$$|S|^{(n-1)(m-1)-1}(|R| - |S|)^n = 1.$$

It means that $m = 2, n = 2$ or either m or n is 1. If $m = n = 2$, then $|R|^2 = |S|^2$ and $|R| = |S| + 1$, which cannot be hold. If $n = 1$, then the graph $TD(R, n)$ has an isolated vertex but $TD(S, m)$ has no isolated vertex.

(ii) If

$$|R|^{n-1} - 1 = |S|^{m-1} - 1.$$

Similar to the proof of Theorem 2.6, we can get $m = n$ and $|R| = |S|$.

□

The next theorem shows that for a field \mathbb{F} , the graph $TD(\mathbb{F}, n)$, can be determined uniquely among all rings.

Theorem 2.9. *Let \mathbb{F} be a finite field and R be a ring. Let m, n be integers. If $TD(\mathbb{F}, n) \simeq TD(R, m)$. Then $m = n$ and $R \simeq \mathbb{F}$.*

Proof. First we prove that R must be a field. On the contrary, assume that R is not a field. Let d_1 and d_2 be two vertex degree of the graph $TD(\mathbb{F}, n)$. Then by Corollary 2.4, we have

$$(3) \quad |d_1 - d_2| \in \{0, 1\}.$$

Let R be a ring which is not a field. Hence there exists a non-zero zero divisor in R , say a . Let b be a non-zero element of R such that $ab = 0$. Obviously, degree of $\mathbf{1} = (1, 1, \dots, 1)$ is $|R|^{m-1} - 2$ if m is divisible by $\text{char}(R)$, and $|R|^{m-1} - 1$, otherwise. Let $\mathbf{a} = (a, a, \dots, a)$. Thus, \mathbf{a} is adjacent to (b_1, \dots, b_n) whenever either $b_1 + \dots + b_n = 0$ or $b_1 + \dots + b_n = b$. Then $\deg(\mathbf{a}) \geq 2|R|^{m-1} - 2$. Obviously, $2|R|^{m-1} - 2 > |R|^{m-1} - 2$, which contradicts Formula (3). The rest of the proof is clear by Theorem 2.6. □

Remark 4. By Corollary 2.4 and Theorem 2.9, we can classify all rings R and integers n , so that the graph $TD(R, n)$ is regular.

3. DOMINATION NUMBER

Let G be a graph. If G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$. It is easy to see that for k -regular graph, $\gamma(G) \geq \frac{n}{k+1}$. The domination number of a graph and its many variations have been extensively studied in the literature [9].

The next result, which is due to Mekiš [15], gives a lower bound for domination number of tensor products of graphs.

Theorem 3.1. [15] *Let G and H be simple graphs. Then*

$$\gamma(G \otimes H) \geq \gamma(G) + \gamma(H) - 1.$$

It is of interest to find the domination number of graphs associated to rings, see [7] and [11]. In this section, we would like to study the domination number of $\gamma(TD(R, n))$. It is easy to see that

$$(4) \quad \gamma(TD(R, n)) \leq \gamma(TD(R, n-1)).$$

In the next theorem we find the domination number of $TD(\mathbb{F}, n)$.

Theorem 3.2. *Let \mathbb{F} be a field with q elements. Let $n > 1$ be an integer. Then*

$$(5) \quad \gamma(TD(\mathbb{F}, n)) = \begin{cases} 2 & \text{if } \mathbb{F} \simeq \mathbb{Z}_2 \text{ and } n = 3, \\ q + 1 & \text{otherwise.} \end{cases}$$

Proof. If $\mathbb{F} \simeq \mathbb{Z}_2$ and $n = 3$, then by Figure (1) one can easily check that $\gamma(TD(\mathbb{F}, n)) = 2$. By Theorem 2.5, we can see that $\gamma(TD(\mathbb{F}, 2)) = q + 1$. Let $D = \{(a, 1, 0, \dots, 0) \mid a \in \mathbb{F}\} \cup \{(1, 0, 0, \dots, 0)\}$. It is fairly easy to see that D is a dominating set. Let $\mathbb{F} \neq \mathbb{Z}_2$ or $n \neq 3$, we prove that $\gamma(TD(\mathbb{F}, n))$ cannot be less than $q + 1$. On the contrary, assume that $D = \{\mathbf{d}_1, \dots, \mathbf{d}_q\}$ is a dominating set for $TD(\mathbb{F}, n)$. By Corollary 2.4, each \mathbf{d}_i can dominate at most q^{n-1} vertices. Obviously, the system of equations

$$\begin{cases} \mathbf{d}_1 \cdot \mathbf{x} = 0 \\ \mathbf{d}_2 \cdot \mathbf{x} = 0 \end{cases}$$

has more than $q^{n-2} - 1 > 1$ non-trivial solutions. Therefore, the set D dominates at most $q(q^{n-1}) - 2$ vertices, which means that D is not a dominating set for $TD(R, n)$. \square

The aforementioned theorem shows that inequality (4) can be strict or can turn into equality.

Remark 5. Let R be a ring which is not a field. Let r be a non-zero non-invertible element of R . Thus the equation $rx + 1 = 0$ has no solution in R . Then $D = \{(a, 1, 0, \dots, 0) \mid a \in R\} \cup \{(1, 0, 0, \dots, 0)\}$ is not a dominating set.

The next two theorems give upper bounds for the domination number.

Theorem 3.3. *Let R be a finite ring which is not a field. Let $n > 1$ be an integer. Then*

$$\gamma(TD(R, n)) \leq |R - U(R)|^2 - 1.$$

Proof. We show that $\{(r, s, 0, \dots, 0) \mid r, s \in R - U(R) \text{ and } (r, s) \neq (0, 0)\}$ is a dominating set for $TD(R, n)$. Let $\mathbf{a} = (a_1, a_2, \dots, a_n) \in R^n$. We have three cases:

- (i) If a_1, a_2 are invertible. Let $z \in Z(R) - \{0\}$. Then \mathbf{a} is adjacent to $(-a_2z, a_1z, 0, \dots, 0)$.
- (ii) If a_1 is invertible but a_2 is not invertible. Hence, there exists $z \in Z(R) - \{0\}$ such that $za_2 = 0$. Then \mathbf{a} is adjacent to $(0, z, 0, \dots, 0)$.
- (iii) If a_1, a_2 both are not invertible. In this case, \mathbf{a} is adjacent to $(-a_2, a_1, \dots, 0)$.

□

Theorem 3.4. *Let R be a finite ring which is not a field. Let $n > 1$ be an integer. Then $\gamma(TD(R, n)) \leq |R - U(R)| + |R| - 2$.*

Proof. Let $A_1 = \{(r, 0, 0, \dots, 0) \mid r \in R - U(R) \text{ and } r \neq 0\}$ and $A_2 = \{(0, s, 0, \dots, 0) \mid s \in R - U(R) \text{ and } s \neq 0\}$ and $A_3 = \{(u, 1, 0, \dots, 0) \mid u \in U(R)\}$. We show that $A_1 \cup A_2 \cup A_3$ is a dominating set for $TD(R, n)$. Let $\mathbf{a} = (a_1, a_2, \dots, a_n) \in R^n$. We have three cases:

- (i) If a_1, a_2 are invertible. Then \mathbf{a} is adjacent to $(-a_2a_1^{-1}, 1, 0, \dots, 0)$.
- (ii) If a_1 is not invertible. Hence, there exists $z \in Z(R) - \{0\}$ such that $za_1 = 0$. Then \mathbf{a} is adjacent to $(z, 0, 0, \dots, 0)$.
- (iii) If a_2 is not invertible. Hence, there exists $z \in Z(R) - \{0\}$ such that $za_2 = 0$. Then \mathbf{a} is adjacent to $(0, z, 0, \dots, 0)$.

□

Finally, we prove that if R is an infinite ring with some restrictions, then the domination number of $TD(R, n)$ is also infinite.

The following well-known lemma is the key for the rest of this section.

Lemma 3.5. *Let V be a vector space over a field \mathbb{F} . If V is written as union of k proper subspaces of V , then $k \geq |\mathbb{F}|$. In particular, if \mathbb{F} is an infinite field, then V cannot be written as union of a finite number of proper subspaces.*

Here $H_{\mathbf{a}}$ denote the hyperplane $\mathbf{a} \cdot \mathbf{x} = 0$.

Theorem 3.6. *Let \mathbb{F} be a field. Then $\gamma(TD(\mathbb{F}, n))$ is finite if and only if \mathbb{F} is a finite field.*

Proof. On the contrary, assume that $D = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ is a dominating set for $TD(\mathbb{F}, n)$. We know that $N(\mathbf{a}_i) \subset H_{\mathbf{a}_i}$. Therefore, $\bigcup_{i=1}^k N(\mathbf{a}_i) \subset \bigcup_{i=1}^k H_{\mathbf{a}_i}$. Then by Lemma 3.5, it follows that \mathbb{F} should be finite. \square

Lemma 3.7. *Let R be a ring and m be a maximal ideal such that R/m has infinitely many elements. Then R^n cannot be written as union of a finite number of proper R -submodules.*

Proof. On the contrary, assume that $R^n = \bigcup_{i=1}^d V_i$, where V_i are R -submodules of R^n . It is known that $R^n \otimes_R R/m$ is a vector space over the field R/m . Then

$$R^n \otimes_R R/m = \left(\bigcup_{i=1}^d V_i \otimes_R R/m \right) = \bigcup_{i=1}^d (V_i \otimes_R R/m).$$

Since R/m is infinite and $V_i \otimes_R R/m$ are vector subspaces, we get contradiction by Lemma 3.5. \square

Theorem 3.8. *Let R be a ring and m be a maximal ideal such that R/m has infinite elements. Then $\gamma(TD(R, n))$ is not finite.*

Proof. The proof is similar to that of Theorem 3.6. \square

Remark 6. Let R be a ring such that $\sup\{|R/m| \mid m \text{ is a maximal ideal of } R\} = \infty$. Then $\gamma(TD(R, n))$ is not finite. Rings \mathbb{Z} and $\mathbb{F}[x]$ are such examples.

More generally, we have the following theorem:

Theorem 3.9. *Let R be a ring. Let $\nu = \sup\{|R/m| \mid m \text{ is a maximal ideal of } R\}$. Then $\gamma(TD(R, n)) \geq \nu$.*

It would be desirable to show that for an arbitrary infinite ring, the domination number is infinite.

4. INDEPENDENCE AND CLIQUE NUMBER

Our aim in this section is to investigate the clique and independence number of $TD(R, n)$. By Theorem 2.5, the next theorem about the clique and independence number for $n = 2$ follows immediately.

Theorem 4.1. *Let \mathbb{F} be a finite field. Then*

$$(6) \quad \omega(TD(\mathbb{F}, 2)) = \begin{cases} |\mathbb{F}| - 1 & \text{if } |\mathbb{F}| \equiv 3 \pmod{4}, \\ 2 & \text{otherwise.} \end{cases}$$

Also, for the independence number we have the following:

$$(7) \quad \alpha(TD(\mathbb{F}, 2)) = \begin{cases} \frac{O(2, \mathbb{F})}{|\mathbb{F}| - 1} + \frac{(|\mathbb{F}|^2 - 1) - O(2, \mathbb{F})}{2} & \text{if } O(\mathbb{F}, 2) \neq 0, \\ \frac{|\mathbb{F}|^2 - 1}{2} & \text{otherwise.} \end{cases}$$

The set $\{e_1, \dots, e_n\}$ is a clique in $TD(R, n)$. Then $\omega(TD(R, n)) \geq n$. In the next theorem, we prove that under some conditions we have equality.

Theorem 4.2. *Let R be an integral domain such that $O(R, n) = 0$. Then $\omega(TD(R, n)) = n$.*

Proof. Let $W = \{a_1, \dots, a_t\}$ be a clique in $TD(R, n)$. We show that a_1, \dots, a_t should be linearly independent over R . Let

$$\alpha_1 a_1 + \dots + \alpha_t a_t = 0.$$

Therefore, by multiplying to a_i for $i = 1, \dots, t$, we have

$$\alpha_i \|a_i\| = 0.$$

Then $\alpha_i = 0$. Since R^n is a free R -module, $t \leq n$. □

If we drop the condition $O(R, n) = 0$, above theorem is no longer hold. The next two theorems show that if $O(R, n) \neq 0$, then the clique number is exponentially large.

Theorem 4.3. *Let \mathbb{F} be a field such that $O(\mathbb{F}, 2) \neq 0$. Then*

$$(8) \quad \omega(TD(\mathbb{F}, n)) \geq |\mathbb{F}|^{\lceil \frac{n}{2} \rceil} - 1.$$

If n is an odd number, then $\omega(TD(\mathbb{F}, n)) \geq |\mathbb{F}|^{\lceil \frac{n}{2} \rceil}$.

Proof. Let $a^2 + b^2 = 0$ and $(a, b) \neq (0, 0)$. Let $\mathbf{a}_i = ae_1 + be_2 + \dots + ae_{2i-1} + be_{2i}$, for $i = 1, \dots, \lceil \frac{n}{2} \rceil$. Let W be the vector subspace generated by $\{\mathbf{a}_i \mid i = 1, \dots, \lceil \frac{n}{2} \rceil\}$. Then $W - \{\mathbf{0}\}$ is a clique in $TD(\mathbb{F}, n)$ of size $|\mathbb{F}|^{\lceil \frac{n}{2} \rceil} - 1$.

If n is odd, then $W \cup \{e_n\}$ is a clique set. Hence, $\omega(TD(\mathbb{F}, n)) \geq |\mathbb{F}|^{\lceil \frac{n}{2} \rceil}$. □

By Theorem 4.1, the inequality (8) can turn into equality for $n = 2$.

Theorem 4.4. *Let R be a ring such that $O(R, 2) \neq 0$. Then*

$$(9) \quad \omega(TD(R, n)) \geq 2^{\lfloor \frac{n}{2} \rfloor} - 1.$$

If n is an odd number, then $\omega(TD(R, n)) \geq 2^{\lfloor \frac{n}{2} \rfloor}$.

Proof. Let $a^2 + b^2 = 0$ and $(a, b) \neq (0, 0)$. Let $\mathbf{a}_i = ae_1 + be_2 + \cdots + ae_{2i-1} + be_{2i}$, for $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$.

Let

$$W = \left\{ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \varepsilon_i \mathbf{a}_i \mid \varepsilon_i \in \{0, 1\} \right\}.$$

Then $W - \{\mathbf{0}\}$ is a clique in $TD(R, n)$ of size $2^{\lfloor \frac{n}{2} \rfloor} - 1$.

If n is odd, then $W \cup \{e_n\}$ is a clique set. Hence, $\omega(TD(R, n)) \geq 2^{\lfloor \frac{n}{2} \rfloor}$. \square

Remark 7. By Proposition 6.1 of [1], one can easily find a better lower bound for the inequality (9).

Remark 8. Let $W = \{\mathbf{a}_1, \dots, \mathbf{a}_t\}$ be a clique set of $TD(R, n)$. Then the set $\Delta = \{(\mathbf{a}_i, 1) \mid i = 1, \dots, t\} \cup \{e_{n+1}\}$ is an independent set for $TD(R, n+1)$. Hence,

$$\omega(TD(R, n)) + 1 \leq \alpha(TD(R, n+1)).$$

Let $W = \{\mathbf{a}_1, \dots, \mathbf{a}_t\}$ be a clique set of $TD(R, n)$ such that $\|\mathbf{a}_i\| = 0$ for $i = 1, \dots, t$. Then the set $\Delta = \{(\mathbf{a}_i, \beta) \mid i = 1, \dots, t \text{ and } \beta \in R^*\} \cup \{\beta e_{n+1} \mid \beta \in R^*\}$ is an independent set for $TD(R, n+1)$. Hence,

$$(|R| - 1)(\omega(TD(R, n)) + 1) \leq \alpha(TD(R, n+1)).$$

The following proposition about tensor product of graphs is straightforward.

Proposition 4.5. *Let G and H be simple graphs. Then $\omega(G \otimes H) = \min\{\omega(G), \omega(H)\}$.*

Definition 1. A clique-loop is a set of pairwise adjacent vertices in a graph, and loop at each vertex. Let us denote by $\overline{\omega}(G)$ the size of the largest clique-loop of G .

The following proposition can be proved easily.

Proposition 4.6. *Let G and H be graphs. Then $\omega(G \otimes H) \geq \overline{\omega}(G \otimes H) = \overline{\omega}(G)\overline{\omega}(H)$*

It is easy to check that $\omega(TD(R, n)) = \omega(\overline{TD(R, n)}) - 1$. However, it seems difficult to find the clique number of $TD(R, n)$, for an arbitrary ring R and integer n .

Theorem 4.7. *Let \mathbb{F} be a field such that $O(\mathbb{F}, 2) \neq 0$. Then $\overline{\omega(TD(\mathbb{F}, n))} = |\mathbb{F}|^{\lceil \frac{n}{2} \rceil}$.*

Proof. Let Δ be a clique-loop of maximum size. We first prove that Δ is a vector subspace. Since Δ is maximum, then $\mathbf{0} \in \Delta$, and if $\mathbf{a}, \mathbf{b} \in \Delta$, then $\mathbf{a} - \mathbf{b} \in \Delta$. Since for all $\mathbf{a}, \mathbf{b} \in \Delta$, we have $\mathbf{a} \cdot \mathbf{b} = 0$, it follows that $\Delta \subseteq \Delta^\perp$. Therefore, $\dim_{\mathbb{F}} \Delta \leq \lceil \frac{n}{2} \rceil$ completes the proof. \square

Now, we will show that the ring R is finite if and only if $\alpha(TD(R, n))$ is finite.

Theorem 4.8. *Let R be an infinite ring. Then $\alpha(TD(R, n)) = \infty$.*

Proof. Let $U(R) = R_1 \cup R_2$ be a partition of invertible elements in such a way that, if $a \in R_i$, then $-a^{-1} \notin R_i - \{a\}$, for $i = 1, 2$. Without restriction of generality, we can assume $|R_1| \geq |R_2|$. Let $\mathfrak{R} := (R - U(R)) \cup R_1$. It means that if x and y are distinct elements of \mathfrak{R} , then $xy \neq -1$. We see at once that $|\mathfrak{R}| = \infty$. Let $\Delta = \{e_1 + ae_2 \mid a \in \mathfrak{R}\}$. Hence, Δ is an independent set with infinitely many elements. \square

5. PLANARITY

In [16], the authors have classified all finite commutative rings R such that $\Gamma(R)$ is planar. In this section, we classify all rings R and n , such that $TD(R, n)$ is planar.

A remarkable characterization of the planar graphs was given by Kuratowski in 1930.

Theorem 5.1. *A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

The next theorem classifies all planar graphs $TD(R, n)$.

Theorem 5.2. *Let R be a commutative ring and n be a natural number. Then $G = TD(R, n)$ is planar if and only if $G = TD(\mathbb{Z}_2, 2)$, $G = TD(\mathbb{Z}_2, 3)$ or $G = TD(\mathbb{Z}_3, 2)$.*

Proof. Let $TD(R, n)$ be a planar graph. Since $\omega(TD(R, n)) \geq n$, we have $n \leq 4$.

Let R be a ring with at least 4 elements. Let a, b, c be three distinct non-zero elements of R .

Therefore, the graph $K_{3,3}$ is a subgraph induced by $\{ae_1, be_1, ce_1, ae_2, be_2, ce_2\}$. Then R is either \mathbb{Z}_2 or \mathbb{Z}_3 , and $n \leq 4$.

Let $n = 4$. It is easy to check that the graph $K_{3,3}$ is a subgraph induced by $\{e_1, e_2, e_1 + e_2, e_3, e_4, e_3 + e_4\}$. Let $G = TD(\mathbb{Z}_3, 3)$. Let H be the subgraph of G induced by $\{e_1, e_2, e_3, 2e_1, 2e_2, 2e_3, e_2 + e_3, e_1 + e_3, e_1 + e_2\}$. Merge $2e_1$ and $e_2 + e_3$, $2e_2$ and $e_1 + e_3$, and $2e_3$ and $e_1 + e_2$. The new graph is isomorphic to the $K_{3,3}$ graph. Then G cannot be planar.

It is easy to check that the graphs $TD(\mathbb{Z}_2, 2)$, $TD(\mathbb{Z}_2, 3)$ and $TD(\mathbb{Z}_3, 2)$ are planar graphs. \square

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